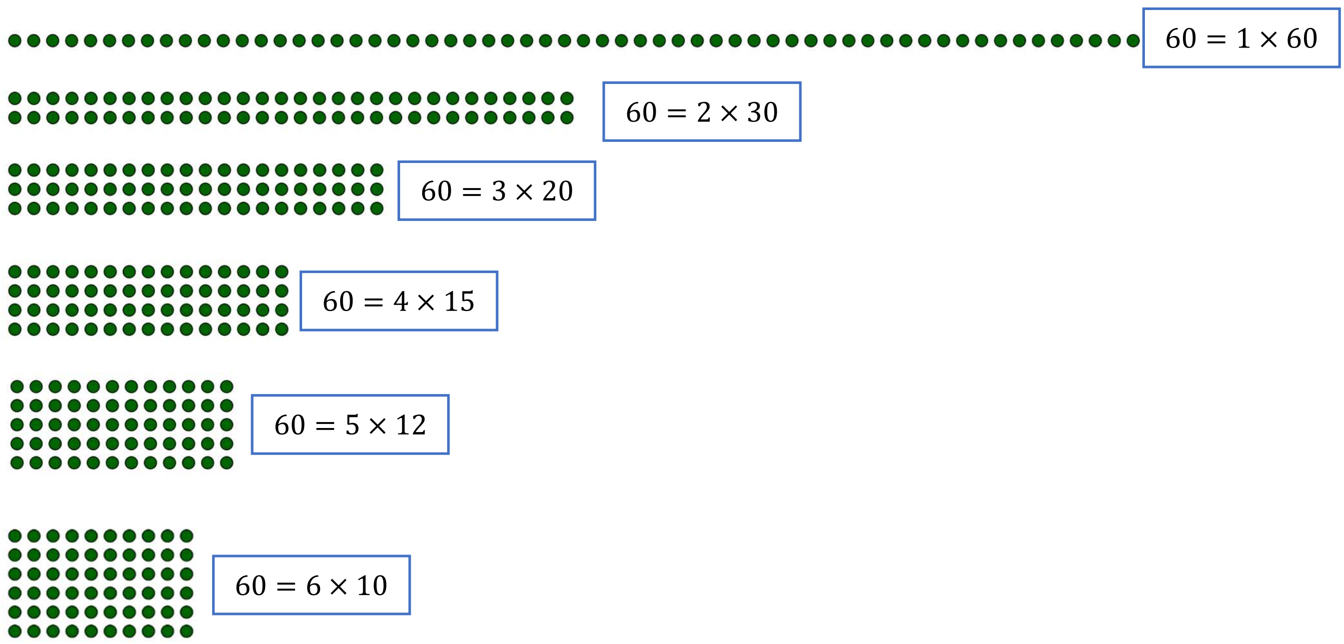
How Many Primes?

Some numbers, like 60, have many factors. This means that you can arrange 60 dots into many different rectangle shapes:



Some numbers, like 13, can't be arranged into many rectangles. In fact, only this (not particularly interesting) rectangle is possible:

Numbers like this are called *prime numbers*.

Prime numbers are numbers with exactly 2 factors.

The full list of the prime numbers starts: $2, 3, 5, 7, 11, 13$ …

How does it continue? Does the list go on forever, or is there such thing as the largest prime?

It might seem that the larger a number gets, the more *likely* it is that it can form many different rectangles. And indeed, in a sense, that's correct.

But we're asking about *existence,* not *likelihood.*

Spoiler alert: I'll reveal the answer on the next page. If you would like to think about the question first, do so now!

Question: How many prime numbers are there?

Answer: There are *infinitely many of them.*

So the primes don't 'run out'. There is no largest prime number.[1]

In the rest of this article, there is a *proof* that there are infinitely many primes. I suggest that you read it slowly and carefully, trying to understand one step at a time.

Proof that there are infinitely many prime numbers

The prime numbers are: $2, 3, 5, 7, \dots$ (and so on)

Let's suppose that there is a biggest prime number, which we'll call $\mathbf{P}$.

So the full list of primes is: $2, 3, 5, 7, \dots, \mathbf{P}$

If we multiply this list of prime numbers together, we get: $2 \times 3 \times 5 \times 7 \times \dots \times \mathbf{P}$

This big number: $(2 \times 3 \times 5 \times 7 \times \dots \times \mathbf{P})$ is a multiple of 2.

It's also a multiple of 3

It's also a multiple of 5

etc. all the way up to: It's also a multiple of $\mathbf{P}$

So the following bigger number: $[(2 \times 3 \times 5 \times 7 \times \dots \times \mathbf{P}) + 1]$ is one more than a multiple of 2. It's also one more than a multiple of 3, and so on.

So it is not a multiple of 2

and it is not a multiple of 3

and it is not a multiple of 5

etc. all the way up to: and it is not a multiple of $\mathbf{P}$

So $[(2 \times 3 \times 5 \times 7 \times \dots \times \mathbf{P}) + 1]$ is not a multiple of any of the prime numbers, so it must itself be prime.

So $[(2 \times 3 \times 5 \times 7 \times \dots \times \mathbf{P}) + 1]$ is a prime number that is bigger than $\mathbf{P}$.

But earlier we said that P was the biggest prime number. Our logic has been correct. So the only thing that could be wrong is our opening statement that there is a biggest prime number. So there can't possibly be a biggest prime number, which means that there are infinitely many prime numbers.

[1] There is the *largest prime number that humans have found so far*. At time of writing, this is $2^{82,589,933} - 1$. This is a very large number. If you wrote it out, you'd need to write down 24,862,048 digits.

How Many Primes?: Comprehension Questions:

1. It's possible to make 6 different rectangles from 60 dots (see the beginning of the article). How many different rectangles can be made with:
   a) 10 dots
   b) 12 dots
   c) 19 dots

2. Which of the numbers of dots from question 1 is prime?

3. Is $(2^{82,589,933} - 1)$ the largest prime number?

How Many Primes?: Comprehension Answers:

1. a) With 10 dots you can make 2 different rectangles:

   b) With 12 dots you can make 3 different rectangles:

   c) With 19 dots you can only make this one rectangle:

2. Which of the numbers of dots from question 1 is prime?
   19 is the prime number. It has two factors, 19 and 1. (And hence only one possible rectangle of dots)

3. Is $(2^{82,589,933} - 1)$ the largest prime number?

   No!

   There are infinitely many prime numbers.