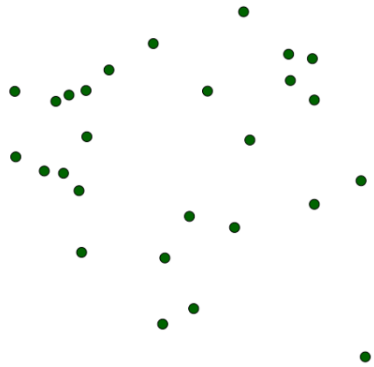
Writing down really big numbers



Humans have developed an efficient and practical way of writing down numbers.

For example, to describe the number of dots in the picture on the right, we can do that in just two characters: 27, meaning 2 lots of 10 plus 7

We also have a well-established symbol for plus, so people understand the number sentence: $10 + 10 + 7 = 27$.

As humans have found the need to describe bigger numbers, notation has been invented to help write these down. Some of these notations will no doubt be familiar to you.

Multiplication ($\times$)

$\times$ can deal with repeated $+$.

If we wanted to write down the number of minutes in a whole day, rather than write

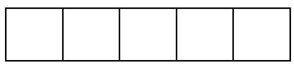
$$\underbrace{60 + 60 + 60 + 60 + 60 + \cdots + 60}_{24 \text{ times}}$$

we can instead write 24×60.

Powers ($\square^{\square}$)

$\square^{\square}$ can deal with repeated $\times$.

For example, $2^5 = 2 \times 2 \times 2 \times 2 \times 2$. This is the number of different ways of colouring these 5 boxes with 2 colours:



Knuth's up-arrow notation (↑)

We can continue this idea of inventing a notation that repeats the previous one. Donald Knuth introduced such a notation, using ↑ as a symbol.

In his system, a single ↑ does the same job as $\square^{\square}$.

For example, $5 \uparrow 3 = 5^3 = 5 \times 5 \times 5 = 125$

A double arrow serves to repeat the single arrow procedure the number of times as indicated by the number on the right.

For example, $5 \uparrow\uparrow 2 = 5 \uparrow 5 = 5^5 = 5 \times 5 \times 5 \times 5 \times 5 = 3125$

and
$$\begin{aligned} 5 \uparrow\uparrow 3 &= 5 \uparrow (5 \uparrow 5) \\ &= 5 \uparrow (3125) \\ &= 5^{3125} \end{aligned}$$

A triple arrow serves to repeat the double arrow procedure, and so on.

I admit that I have never found a need to use Knuth's arrow notation because I haven't needed to consider a number large enough to make it convenient, but some humans have; there is a number called *Graham's Number*, which arises in part of an answer to a certain Maths problem that is so big that it's helpful to have a notation even more efficient than Knuth's arrow notation in order to write it down.

Writing down really big numbers: Comprehension Questions:

1. Write out 5×17 in terms of repeatedly adding $17s$ (You don't have to calculate it)

2. Write out 17^5 in terms of repeatedly multiplying $17s$ (You don't have to calculate it)

3. Work out the value of the following:

    a. $3 \uparrow 2$

    b. $2 \uparrow 3$

    c. $1 \uparrow 98$

4. Which one of the following numbers is equal to $2 \uparrow (2 \uparrow 4)$?

    a) 2^6 b) 2^8 c) 4^4 d) 2^{16}

5. Which of the following numbers are equal to $3 \uparrow\uparrow 4$?
(there may be more than one correct answer!)

a) 3^4 b) $3 \uparrow (3 \uparrow (3 \uparrow 3)))$ c) $3^{\left(3^{\left(3^3\right)}\right)}$ d) $3^{\left(3^{27}\right)}$ e) $4 \uparrow (4 \uparrow 4)$

6. Which of the two numbers $10 \uparrow\uparrow 2$ and $2 \uparrow\uparrow 10$ is bigger?

Writing down really big numbers: Comprehension Answers:

1. $5 \times 17 = 17 + 17 + 17 + 17 + 17$

2. Write out $17^5 = 17 \times 17 \times 17 \times 17 \times 17$

3.

   a. $3 \uparrow 2 = 3^2 = 9$

   b. $2 \uparrow 3 = 2^3 = 8$

   c. $1 \uparrow 98 = 1^{98} = 1$

4. Which one of the following numbers is equal to $2 \uparrow (2 \uparrow 4)$?

   $2 \uparrow (2 \uparrow 4) = 2 \uparrow (2^4) = 2 \uparrow 16 = 2^{16}$, so the answer is d) 2^{16}

5. Which of the following numbers are equal to $3 \uparrow\uparrow 4$?

$$\begin{aligned} 3 \uparrow\uparrow 4 &= 3 \uparrow (3 \uparrow (3 \uparrow 3)) \\ &= 3 \uparrow (3 \uparrow 3^3) \\ &= 3 \uparrow 3^{(3^3)} \\ &= 3^{\left(3^{(3^3)}\right)} \\ &= 3^{(3^{27})} \end{aligned}$$

so the answers are: b) $3 \uparrow (3 \uparrow (3 \uparrow 3))$ c) $3^{\left(3^{(3^3)}\right)}$ d) $3^{(3^{27})}$

6. Which of the two numbers $10 \uparrow\uparrow 2$ and $2 \uparrow\uparrow 10$ is bigger?

   $10 \uparrow\uparrow 2 = 10 \uparrow 10 = 10^{10}$

$$2 \uparrow\uparrow 10 = 2 \uparrow \left(2 \uparrow \left(2 \uparrow \left(2 \uparrow \left(2 \uparrow \left(2 \uparrow \left(2 \uparrow \left(2 \uparrow \left(2 \uparrow 2 \right) \right) \right) \right) \right) \right) \right) \right)$$

$$= 2^{2^{2^{2^{2^{2^{2^{2^{2^{2}}}}}}}}}$$

   $2 \uparrow\uparrow 10$ is much bigger than $10 \uparrow\uparrow 2$

   because (and there are many other arguments that would work):

$$10^{10} < 16^{16} = (2^4)^{16} = 2^{(2^6)} < 2^{(2^{16})} = 2^{2^{2^{2}}} < 2^{2^{2^{2^{2^{2^{2^{2^{2^{2}}}}}}}}}$$