Deficient, Perfect and Abundant Numbers

The ideas explored in this article are ideas that were explored by the Ancient Greeks over 2000 years ago, and have been studied further since, notably in the 1600s and 1800s. They may, perhaps, be pretty useless ideas for the world, but are quite fun, and despite leading mathematicians playing with the ideas, mysteries remain, as I shall explain.

Some numbers, like 12, have many factors.

> (A *factor* of a number is a number that allows exact division. For example, 3 is a *factor* of 12, because there exists a whole number that fills the gap in this number sentence: $3 \times \square = 12$.)

Some numbers, like 13, are prime and have just two. (In fact there are *infinitely* many of these prime numbers; that's another story for another day).

> (A *prime* number is a number with exactly two factors. For example, 13 is prime, because it has exactly two factors, 1 and 13.)

We can get a sense of how 'well-endowed' a number is with factors, taking into account its size (after all, bigger numbers have a higher chance of having factors, in some sense, than smaller numbers), by considering the sum of all its factors less than itself, and comparing this 'factor sum' to the number itself.

For example, consider the number 14. Its factors are $1, 2, 7, 14$. If we add together all the factors lower than 14 itself, we get $1 + 2 + 7 = 10$. This is less than 14, so we say that 14 is *deficient.*

For example, consider the number 12. Its factors are $1, 2, 3, 4, 6, 12$ If we add together all the factors lower than 12 itself, we get $1 + 2 + 3 + 4 + 6 = 16$. This is more than 12, so we say that 12 is *abundant.*

If we manage to find a number that is neither deficient nor abundant, so the sum of all its factors less than itself is equal to the number itself, this number is described as *perfect.*

The smallest perfect number is 6.

(The sum of the factors of 6, excluding 6 itself, is $1 + 2 + 3 = 6$)

This table shows which of the three categories each of the first 14 numbers belongs to:

number	sum of factors not including itself	Deficient , Perfect or Abundant?
1	0	Deficient
2	1	Deficient
3	1	Deficient
4	3	Deficient
5	1	Deficient
6	6	Perfect
7	1	Deficient
8	7	Deficient
9	4	Deficient
10	8	Deficient
11	1	Deficient
12	16	Abundant
13	1	Deficient
14	10	Deficient

If we continued the table, would there be any more perfect numbers than just 6? Would *deficient* continue to dominate so much?

All prime numbers are deficient, and there are infinitely many of these, so there are infinitely many deficient numbers.

All multiples of 12 are abundant numbers. For a proof of this, let $12n$ be any multiple of 12. (by $12n$, I mean $12 \times n$) Then $n, 2n, 3n, 4n$ and $6n$ are all factors of $12n$ that are lower than $12n$. If we add these up, we get $n + 2n + 3n + 4n + 6n = 16n$, which is larger than $12n$, so $12n$ is abundant. So all multiples of 12 are abundant.

There are infinitely many multiples of 12, so there are infinitely many abundant numbers.

So there are infinitely many deficient numbers, and infinitely many abundant numbers, but are there infinitely many perfect numbers? Fascinatingly, **nobody in the world knows**. At time of writing, there are 47 perfect numbers known, and they're all even. Again, **nobody knows** whether this is a coincidence, or whether it's impossible to get an odd perfect number.

Perhaps you will be the first person in human history to solve one of these mysteries.

Deficient, Perfect and Abundant Numbers: Comprehension Questions:

1. Write down all the factors of 15
2. Find out whether 15 is abundant, perfect or deficient.
3. In this table of values, there are many numbers that have 1 in the second column. What type of numbers are these?

   Will these types of numbers always have value 1 in the second column, and hence always be deficient?

number	sum of factors not including itself	Abundant, Perfect or Deficient?
1	0	Deficient
2	1	Deficient
3	1	Deficient
4	3	Deficient
5	1	Deficient
6	6	Perfect
7	1	Deficient
8	7	Deficient
9	4	Deficient
10	8	Deficient
11	1	Deficient
12	16	Abundant
13	1	Deficient
14	10	Deficient

4. One of the five numbers: $26, 27, 28, 29, 30$ is perfect. Find out which it is.
5. Find out whether 945 is abundant, perfect or deficient.

Deficient, Perfect and Abundant Numbers: Comprehension Answers:

1. The factors of 15 are $1, 3, 5, 15$.

2. $1 + 3 + 5 = 9 < 15$, so 15 is deficient.

3. They are prime numbers. Primes will always have a 1 in the second column. This is because a prime's factors are itself and 1, so the sum of factors less than itself will be 1.

4. 28 is the perfect number:

number	sum of factors not including itself	Abundant, Perfect or Deficient?
26	16	Deficient
27	13	Deficient
28	28	Perfect
29	1	Deficient
30	42	Abundant

5. 945 is abundant (Sum of factors not including itself $= 1 + 3 + 5 + 7 + 9 + 15 + 21 + 27 + 35 + 45 + 63 + 105 + 135 + 189 + 315 = 975 > 945$)