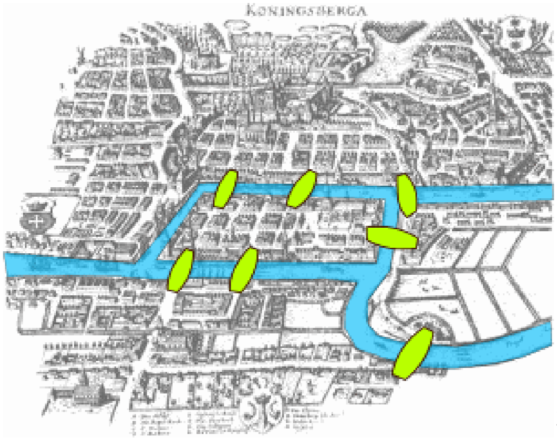
The 7 Bridges of Königsberg

This article is all about a puzzle that was motivated by a real life situation, yet is said to have inspired the development of the abstract mathematics of *graph theory* and *topology*.

[1]

Königsberg was a port city on the south eastern corner of the Baltic Sea. It was damaged by Allied bombing in 1944 and during the Battle of Königsberg in 1945; it was then captured and was given to the Soviet Union on 9 April 1945. It is today known as Kaliningrad and remains under Russian control.

Here is a map of Königsberg as it was in the 1730s. The river and the bridges are highlighted because the Bridges of Königsberg puzzle is about these bridges.


[2]

The puzzle is this:

> Is it possible to find a walk in Königsberg that would cross each of the 7 bridges once and only once*?

(*We are assuming that the only way to cross the river is by using a bridge; no swimming allowed. Also, we are staying in the city of Königsberg, so we're not allowing a solution that involves going round the world, for example.)

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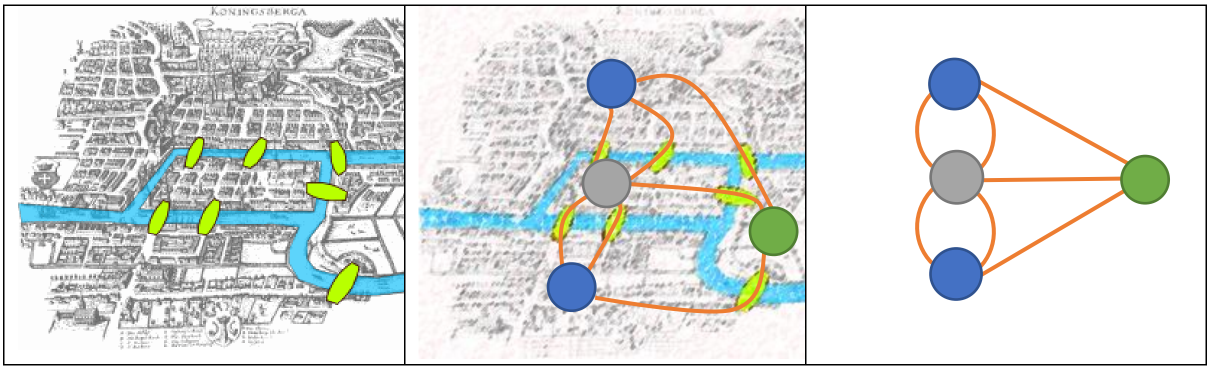
[1] This picture was taken from Google Maps on 28 June 2019

[2] This picture was copied from the Wikipedia page: *Seven Bridges of Königsberg.* Retrieved on 28 June 2019.

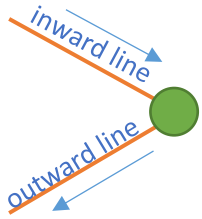
If this is the first time you've seen this puzzle, try for a couple of minutes to find an answer!

Often the first step in using Maths to try to solve a real world problem, a good first move is to produce an abstract version of the problem to make for easier work. We can do this here:



So, looking at the diagram on the right, we are searching for a path between the colourful dots, and we have to travel along each of the orange lines exactly once.

A solution to this puzzle (that is, a *proof* that it is fact impossible) is attributed to prolific mathematician, Leonhard Euler, whose argument went as follows:



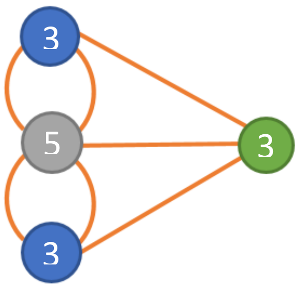
In order not to go over the same orange line twice, when we travel along a line to get to a dot, we must then leave the dot by a different line.

This means that, apart from the dots at the start or end of the journey, we must travel through the dots 'in through one line, out through another'.

We might travel through the same *dot* several times, but the number of inward lines to this dot must equal the number of outward lines from this dot.

So, if we were able to travel along each orange line exactly once, there must be an **even number of lines attached to every dot that's not the starting point or end point of the walk.**

We can now analyse the *number of lines coming from each dot* in the Königsberg diagram to see whether this is possible:



We see that there aren't any dots at all in which the number of lines attached is even.

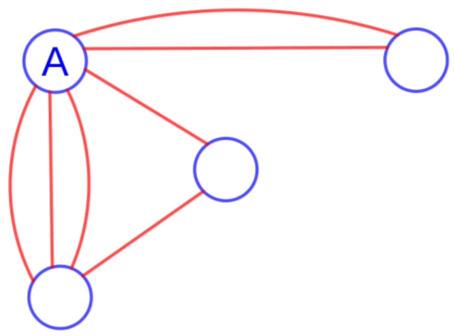
This means that all four dots must be either a start point or an end point of the walk.

But this is not possible; only 2 dots can be.

So, without resorting to miles of walking to 'test' every possible journey, Euler had a convincing argument (a *proof*) that it is **not possible** to find a walk in Königsberg that would cross each of the 7 bridges once and only once

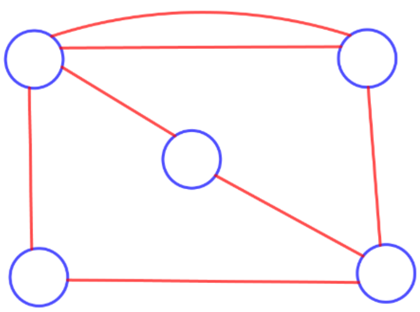
The 7 Bridges of Königsberg: Comprehension Questions:

1. In the graph below, how many lines are connected to the dot marked A?



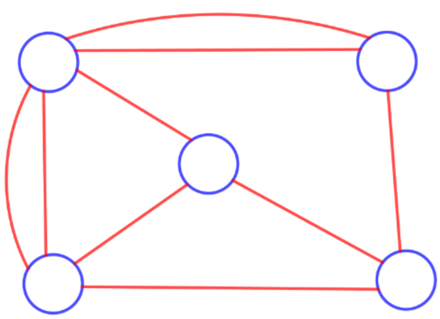
2. a) Copy the graph on the right and fill in each dot with the number of edges attached to it.

   b) Find a path between the dots that involves travelling along each of the red lines exactly once.



3. a) Copy the graph on the left and fill in each dot with the number of edges attached to it.

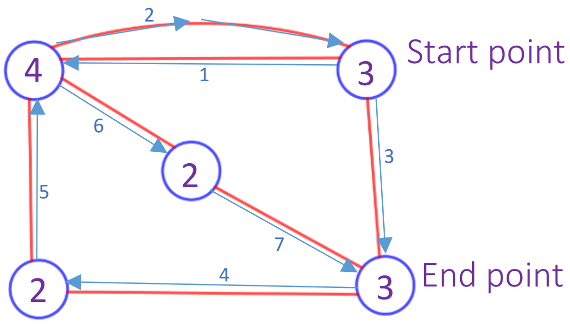
   b) Is it possible to find a path between the dots that involves travelling along each of the red lines exactly once? If so, find it. If not, explain why not.



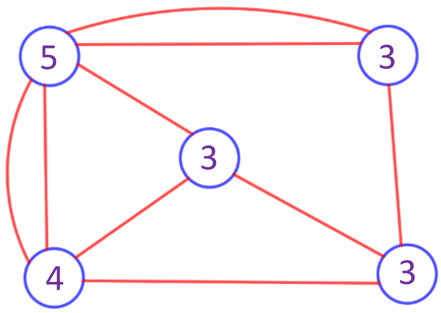
The 7 Bridges of Königsberg: Comprehension Answers:

1. 6 dots

2. A possible path is shown below with the arrows and the blue numbers from 1 to 7. There are many other correct answers for the path, so chances are you've found a different one to the one I display here. You might like to figure out how many correct answers there are:



3. a)



b) It is not possible to find a path between the dots that involves travelling along each of the red lines exactly once. This is because there are too many dots with an odd number of edges connected to it. (There are four of them, which is greater than two.)