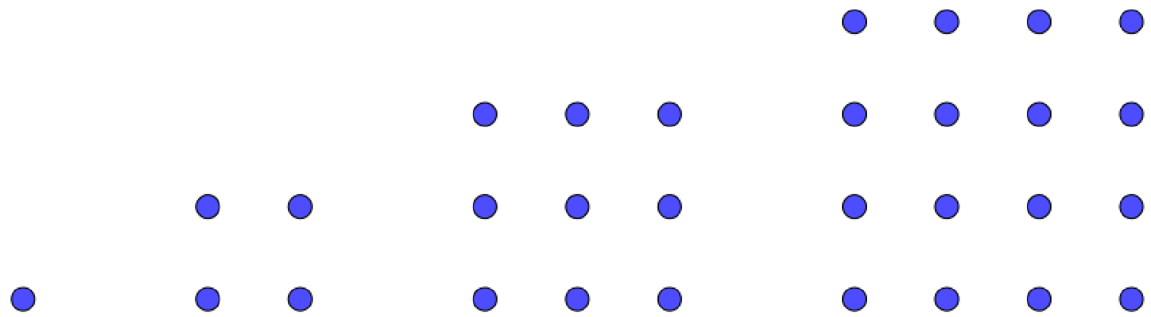
Sum of Two Squares

The square numbers are $1, 4, 9, 16, 25, 36, \ldots$

In other words, they are $1 \times 1,\ 2 \times 2,\ 3 \times 3,\ 4 \times 4$ etc. as illustrated here:



$1^2 = 1$ $2^2 = 4$ $3^2 = 9$ $4^2 = 16$

You can write the number 34 as the some of two square numbers, like this:

$34 = 3^2 + 5^2$.

Fill in the grid below to write as many of the given numbers as the sum of two squares as you can (You are allowed to use 0, because $0 = 0^2$). 34 has been done for you.

1 =	2 =	3 =	4 =
5 =	6 =	7 =	8 =
9 =	10 =	11 =	12 =
13 =	14 =	15 =	16 =
17 =	18 =	19 =	20 =
21 =	22 =	23 =	24 =
25 =	26 =	27 =	28 =
29 =	30 =	31 =	32 =
33 =	34 = $3^2 + 5^2$	35 =	36 =
37 =	38 =	39 =	40 =

Does there seem to be a pattern for the numbers that you aren't able to write as the sum of two squares?

Sum of Two Squares - Answers and Further Thoughts

$1 = 1^2 + 0^2$	$2 = 1^2 + 1^2$	$3 =$	$4 = 2^2 + 2^2$
$5 = 1^2 + 2^2$	$6 =$	$7 =$	$8 = 2^2 + 2^2$
$9 = 0^2 + 3^2$	$10 = 1^2 + 3^2$	$11 =$	$12 =$
$13 = 2^2 + 3^2$	$14 =$	$15 =$	$16 = 0^2 + 4^2$
$17 = 1^2 + 4^2$	$18 = 3^2 + 3^2$	$19 =$	$20 = 2^2 + 4^2$
$21 =$	$22 =$	$23 =$	$24 =$
$25 = 0^2 + 5^2$	$26 = 1^2 + 5^2$	$27 =$	$28 =$
$29 = 2^2 + 5^2$	$30 =$	$31 =$	$32 = 4^2 + 4^2$
$33 =$	$34 = 3^2 + 5^2$	$35 =$	$36 = 0^2 + 6^2$
$37 = 1^2 + 6^2$	$38 =$	$39 =$	$40 = 2^2 + 6^2$

Well done if you found enough of these that you discovered that it *seemed* impossible to write any of the numbers in the 3rd column as the sum of two squares. There's an outline of a proof that this is impossible, below:

Proof that the sum of two square numbers can't be $\mathbf{3}$ more than a multiple of $\mathbf{4}$.

All positive whole numbers are either even (Case 1) or odd (Case 2)

Case 1 Suppose we're given an even number. This even number is $2 \times n$ for some whole number n. If we square this, we get $2 \times n \times 2 \times n$, which is the same as $4 \times n^2$, which is a multiple of 4.

Case 2 Suppose we're given an odd number. This odd number is $2 \times n + 1$ for some whole number n. If we square this, we get $(2 \times n + 1) \times (2 \times n + 1)$, which is the same as $4 \times (n^2 + n) + 1$, which is 1 more than a multiple of 4.

So a square number is either a multiple of 4 or 1 more than a multiple of 4.

This means that if we add two square numbers together, we'll get either a multiple of 4, or 1 more than a multiple of 4, or 2 more than a multiple of 4, but never 3 more than a multiple of 4.

As for the other 'impossible' ones, the second column is quite interesting, as it seems that every other number in the column is impossible. Maybe it's because all of these are equal to one of the numbers in the third column doubled.

Then what about the others that are highlighted? Maybe it's all multiples of numbers in the third column that don't work. But then seeing that 9 is possible ruins this idea.
There is a theorem, called the *Sum of two squares theorem* that reveals the answer. If you're curious, t ry to look this up and understand it!

To find out more, look up
Sum of Two Squares Theorem