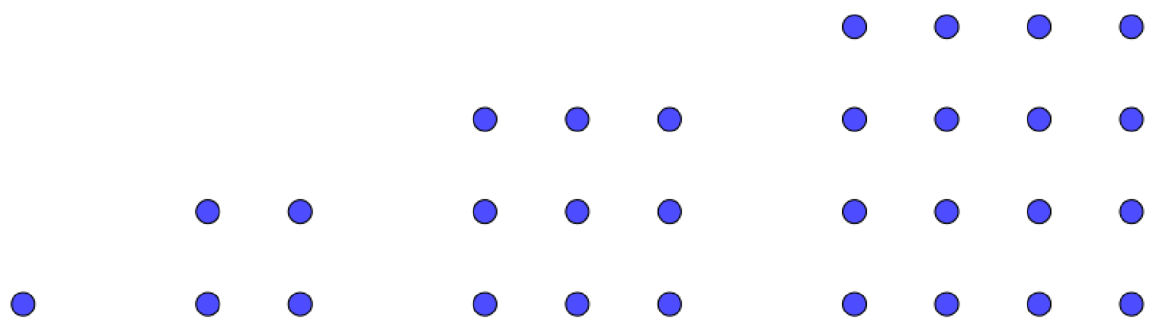
Sum of Some Squares

The square numbers are $1, 4, 9, 16, 25, 36, \ldots$

In other words, they are $1 \times 1,\ \ 2 \times 2,\ \ 3 \times 3,\ \ 4 \times 4$ etc. as illustrated here:



$1^2 = 1$ $\quad$ $2^2 = 4$ $\quad$ $3^2 = 9$ $\quad$ $4^2 = 16$

You can write the number 6 as the sum of three square numbers, like this:

$6 = 2^2 + 1^2 + 1^2$.

1. Write as many of the numbers in the grid below as the sum of square numbers. (6 has been done for you). Your challenge is to do each in **no more than four square numbers**.

1 =	2 =	3 =	4 =
5 =	$6 = 2^2 + 1^2 + 1^2$	7 =	8 =
9 =	10 =	11 =	12 =
$13 =$	$14 =$	$15 =$	$16 =$

2. Now try these larger numbers. Again, the challenge is to write them as the sum of **no more than four square numbers**.

$23 =$
$47 =$
$79 =$
$247 =$

Sum of Some Squares - Answers and Further Thoughts

1.

$1 = 1^2$	$2 = 1^2 + 1^2$	$3 = 1^2 + 1^2 + 1^2$	$4 = 2^2$
$5 = 1^2 + 2^2$	$6 = 2^2 + 1^2 + 1^2$	$7 = 2^2 + 1^2 + 1^2 + 1^2$	$8 = 2^2 + 2^2$
$9 = 0^2 + 3^2$	$10 = 1^2 + 3^2$	$11 = 3^2 + 1^2 + 1^2$	$12 = 2^2 + 2^2 + 2^2$
$13 = 2^2 + 3^2$	$14 = 3^2 + 2^2 + 1^2$	$15 = 3^2 + 2^2 + 1^2 + 1^2$	$16 = 4^2$

2.

$23 = 3^2 + 3^2 + 2^2 + 1^2$
$47 = 5^2 + 3^2 + 3^2 + 2^2$
$79 = 6^2 + 5^2 + 3^2 + 3^2$
$247 = 9^2 + 9^2 + 9^2 + 2^2$

You might have noticed that the numbers I chose in question 2 each needed the full four squares. This is something to do with the fact that they are each 7 more than a multiple of 8
($23 = 2 \times 8 + 7$, $47 = 5 \times 8 + 7$, $79 = 9 \times 8 + 7$, $240 = 30 \times 8 + 7$).

So we found that we can write the first 16 whole numbers as the sum of no more than 4 squares.

What about all the rest of the whole numbers?

If someone thinks of a number, can be sure that we can write that number as the sum of at most 4 square numbers?

Well, as it happens, we do *know for sure* that we'll always be able to do this. You may be thinking: How can we possibly be sure? There are infinitely many numbers to test, after all.

That's the power of a mathematical *theorem*. In this case, it's called the Lagrange's Four-Square Theorem. Lagrange was a mathematician of Italian and French descent. He *proved* the Four-Square Theorem in 1770.

There is a theorem like this for cubes. I invite you to experiment to guess how many cube numbers we need if we want to write numbers as the sum of cubes (it was 4 for squares).

(There are also theorems like this for 4th powers, 5th powers, etc.)

To find out more, look up
Lagrange
Fermat polygonal number theorem
Legendre's three-square theorem
or investigate:
writing numbers as the sum of cube numbers. Whereas 4 squares were needed, how many cubes are needed?